**The Ising Model (Introduction)**

In 1925, Ernst Ising in his PhD thesis proposed the Ising model, (in its one-dimensional version). At that time, it was proposed as a tool to describe the thermodynamic properties of magnetic systems, from the microscopic point of view.

In the case where system does not exhibit any phase transition, for T > 0, Ising found and incorrectly concluded that the whole model was not useful to describe such systems.

However, this model has been later studied again in different configurations. And from the studies, many important properties have been discovered. Historically, Ising model has been one of the most heavily studied model in the field of statistical mechanics. This model was also often used as a testing ground, when new theories or methods are developed.

Another extremely important characteristic of the Ising model, is that it does not only apply to magnetic systems, but can also be applied to many other systems, which can be shown to be equivalent to an appropriately defined Ising model.

The d-dimensional Ising model is defined as follows - consider a d-dimensional lattice with N sites, each labelled by the index i = 1 , … , N.

In general case of d-dimentions, the lattice is supposed to be hypercubic, but this is not necessary. In 2-dimensions, we can consider “triangular” or “honeycomb” lattices, while in 3-dimensions, we can have “body-centered” or “face-centered” cubic lattices.

What distinguishes one lattice from another is its coordination number z. The coordination number (z) is defined as the number of the nearest neighbours of a site.

In the case of hypercubic lattices, it can be easily seen that

z = 2d

, where d is the dimensionality of the system.

The discrete variables Si are the degrees of freedom of the model, which are defined on each site that can only assume the values +1 and -1 . As N are the total number of sites, and each site has 2 possible configurations. Therefore, the number of the possible configurations of the system is 2N .

The lattice in the Ising model actually represents the atomic lattice of a metal and the variables Si are the spins of the atoms, or maybe their component along the vertical axis.

Therefore,

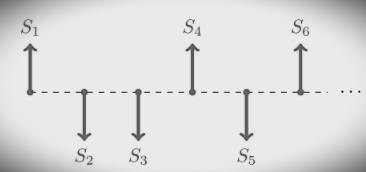
Si = +1 🡪 corresponds to a spin pointing upwards,

Si = -1 🡪 corresponds to a spin pointing downwards.

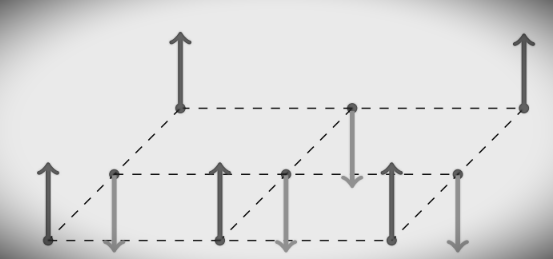
The study of this model determines that, whether all these spins can align so that the system can have a spontaneous net magnetization, and also how this happens.

However, as the Ising model can also be used to describe completely different systems, this interpretation is not the only possible interpretation. Even though other representations might have many similarities, since this model is always been associated to magnets, so in the following contexts, terminology proper only to magnetic systems are used mostly.

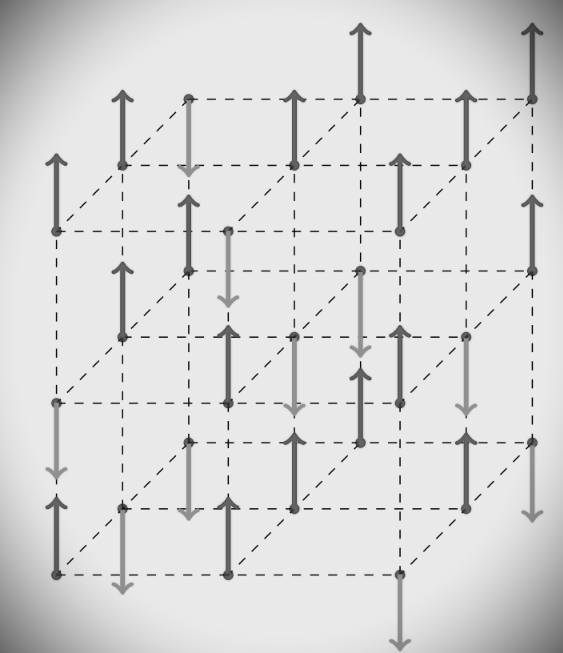
The visual representation of Ising model for various dimension is as follows :-



*Fig : Ising model in 1-dimension*



*Fig : Ising model in 2-dimensions*



*Fig : Ising model in 3-dimensions*

The usefulness of the Ising model goes much further than what can now be imagined, as this model can also be used to describe systems different from magnetic ones. Lattice theories are in fact widely used in many areas of physics. For example - apart from general applications of Ising model in solid state physics, it is also used in formulating quantum relativistic theories in terms of lattices. We therefore see that this also models with dimensionality greater than 3, which can be actually useful.

For making this model look interesting, the degrees of freedom Si must not be independent. We therefore assume that the spins interact with each other with exchange interactions that couple in general an arbitrary number of spins, and also with an external field H that can change from site to site. Therefore, the most general form of the Hamiltonian of Ising model for a given spin configuration is :

First term, having expression like Si shows the dependence upon external magnetic field (Hi). Second term, having expression like S­iSj shows the correlation between neighbouring spins in 1-dimentional system. Similarly, third term having expression like SiSjSk shows the correlation between spins in 2-dimentional system, as one of the spin is reference spin, and other two are it’s neighbours along the 2-dimentions. Similarly, further terms keep on adding till the correlation term due to d-dimentional system. (where d = 1, 2, .. N)

The first two negative signs symbolise the alignment of two adjacent magnetic moments, in general. So for the stability of system, considering energetic convenience, it is better to have as many aligned spins as possible.

There are many methods to find the solution of Ising model Hamiltonian, some of them are :-

1. Transfer Matrix method
2. Mean field theory

**Transfer Matrix method**

The Transfer Matrix method allows us to extend our considerations also when h != 0 (having some external magnetic field) and to compute other interesting properties is the so called transfer matrix method, which basically consists in defining an appropriate matrix related to the model such that all the thermodynamic properties of the system can be extracted from the eigenvalues of this matrix. We are going to see this method applied to the one-dimensional Ising model, but its validity is completely general; we will stress every time if we are stating general properties of the transfer matrix method or restricting to particular cases.

For one-dimensional Ising model, the Hamiltonian with periodic boundary conditions when an external field is present is such that :

Where, : is called reduced Hamiltonian.

Now, the Partition function for this is :-

For a 1-dimensional system of N-spins, Partition function is :-